Mean-convex Smoothing of Singular Boundaries

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Calculus of Variations on the Lake Frauenchiemsee

Minimal hypersurfaces

Let $(\Sigma_t)_{|t|<\varepsilon}$ be a smooth family of hypersurfaces with $\Sigma_0 = \Sigma$ and normal velocity $f\nu$ at $t = 0$. Then

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Area}(\Sigma_t) = \int_\Sigma Hf \, d\mu.$$

Thus Σ is stationary precisely when $H = 0$. A smooth stationary hypersurface is also called minimal.
At a minimal hypersurface, the second variation defines the quadratic form

$$Q_\Sigma(f, f) := \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Area}(\Sigma_t).$$

It is *stable* if $Q_\Sigma(f, f) \geq 0$ for every compactly supported f.
An area-minimizing hypersurface satisfies

$$\mathrm{Area}(\Sigma) \leq \mathrm{Area}(\Sigma')$$

for every compactly supported competitor Σ'. Hence every area minimizer is stable, and every stable hypersurface is minimal.

The Plateau problem

Given a smooth closed $(m-1)$-dimensional submanifold $\Gamma \subset \mathbb{R}^{m+1}$, one seeks a hypersurface spanning Γ with least area:

$$\mathrm{Area}(\Sigma) = \inf\left\{\mathrm{Area}(\Sigma') : \partial\Sigma' = \Gamma\right\}.$$

Existence is obtained by allowing generalized hypersurfaces. Regularity theory asks whether the resulting minimizer is actually smooth.
In the interior, codimension-one area minimizers satisfy

$$\mathrm{Sing}(\Sigma) = \varnothing \quad \text{if } m \le 6, \qquad \dim_{\mathcal{H}} \mathrm{Sing}(\Sigma) \le m - 7 \quad \text{if } m \ge 7.$$

The threshold is sharp. Bombieri–De Giorgi–Giusti proved that the Simons cone

$$C_{\mathrm{S}} := \left\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\right\} \subset \mathbb{R}^8$$

is area-minimizing. It is singular at the origin.
Thus singularities occur naturally.

Smooth area minimizers

For a (singular) area minimizer Σ, one may ask whether it can be approximated by smooth area minimizers:

$$\Sigma_j \text{ smooth and area-minimizing}, \qquad d_H(\Sigma_j, \Sigma) \longrightarrow 0.$$

In 1985, **Hardt–Simon** constructed smooth one-sided area minimizers asymptotic to every area-minimizing cone C with an isolated singularity. One can also perturb the data of the minimizing problem:

$$\Gamma_j \longrightarrow \Gamma \quad \text{or} \quad g_j \longrightarrow g, \qquad \Sigma_j \text{ smooth and area-minimizing.}$$

In ambient dimension 8, Hardt–Simon perturbed Γ, while **Smale (1993)** perturbed g. **Chodosh–Mantoulidis–Schulze (2023)** extended both results to dimensions 9 and 10, and
Chodosh–Mantoulidis–Schulze–Wang (2025) to dimension 11.

Lawson's question

Lawson (1986, Proc. Sympos. Pure Math.). If $C \subset \mathbb{R}^{n+1}$ is a stable minimal cone, does every $\varepsilon > 0$ admit a properly embedded smooth hypersurface $S_\varepsilon \subset B_1$ such that

$$H_{S_\varepsilon} > 0, \qquad d_H(S_\varepsilon, C \cap B_1) < \varepsilon?$$

Thus Lawson asked only for a local smoothing inside B_1.
Zhihan Wang (2024, GAFA). Writing $C = \partial E$, Wang's construction is global: there is a properly embedded smooth hypersurface

$$S \subset \mathrm{Int}(E)$$

asymptotic to C at infinity. More precisely,

- if C is minimizing in E, then S is area-minimizing;
- otherwise, S is a strictly mean-convex self-expander.

A final local perturbation in the minimizing case gives Lawson's strict inequality.

Quasi-regular boundaries

Let $\Omega \Subset (M^n, g)$, set $\Sigma := \partial\Omega$, and let $\rho(x) := \mathrm{dist}_g(x, \Sigma)$. For $r > 0$, define

$$\begin{aligned} \Lambda_r &:= \big\{ p \in \partial\Omega : \text{ there is } x \in \Omega \text{ with } d_g(x, p) = \rho(x) = r \big\}, \\ \Lambda &:= \bigcup_{r>0} \Lambda_r. \end{aligned}$$

Thus Λ is precisely the part of the boundary touched by an interior tangent ball.
We call $\partial\Omega$ C^2-*quasi-regular* if Λ is a C^2 hypersurface near which Ω lies on one side, and if, for every $r_0 > 0$,

$$\bigcup_{r>r_0} \Lambda_r \quad \text{is relatively open in } \partial\Omega.$$

No regularity is imposed on $\partial\Omega \setminus \Lambda$.

Stationary boundaries are quasi-regular

Suppose that $\partial\Omega$ is stationary and let $p \in \Lambda$. After blow-up, an interior tangent ball at p gives

$$\operatorname{spt} C_p \subset \{x_n \geq 0\}$$

for every tangent cone C_p.

The coordinate function x_n is weakly harmonic on the stationary cone C_p.

Since $x_n \geq 0$ and vanishes at the vertex, the mean-value inequality gives

$$x_n \equiv 0 \quad \text{on } C_p, \qquad \text{hence} \qquad C_p = \{x_n = 0\}.$$

Allard regularity therefore gives

$$p \in \mathcal{R}(\partial\Omega).$$

Thus $\Lambda \subset \mathcal{R}(\partial\Omega)$, and the local normal geometry of the regular part shows that $\partial\Omega$ is C^2-quasi-regular.

The same argument applies to prescribed-mean-curvature boundaries, since their rescaled mean curvature tends to zero.

See Simon (1983, Theorem 36.5 and Remark 36.6) and Ilmanen (1996, Lemma 10).

Gromov's mean-convex smoothing theorem

Let $\Omega \Subset (M^n, g)$ have compact C^2-quasi-regular boundary. If

$$\delta \in \mathbb{R}, \qquad H_\Lambda \geq \delta$$

with respect to the outward normal, then there exist smooth open sets $\Omega_j \Subset \Omega$ such that

$$d_H(\partial\Omega_j, \partial\Omega) \longrightarrow 0, \qquad H_{\partial\Omega_j} \geq \delta - o(1).$$

The theorem in this form was formulated by **Gromov (2014)**. We present two proofs.

- **Our proof.** Rifford's Sard theorem and Federer's theory of positive reach give a $C^{1,1}$ approximation.
- **Gromov's first proof.** Bend the distance function, smooth it, and take a level set.

We found our proof in 2025, before realizing the relevance of Gromov's paper. Gromov sketches two different arguments; we understood his first argument two weeks ago.

M. Gromov, *Plateau–Stein Manifolds* (2014), Sections 3.2–3.4.

A regular distance level has positive reach

A closed set $E \subset \Omega$ has *positive reach in* Ω if every point of Ω sufficiently close to E has a unique nearest point in E.
Let

$$d_\Sigma = \mathrm{dist}(\,\cdot\,,\Sigma), \qquad F_s = \{d_\Sigma \geq s\}.$$

Every minimizing segment from an interior point ends at Λ. Hence

$$d_\Sigma = \mathrm{dist}(\,\cdot\,,\Lambda) \qquad \text{in } \Omega.$$

Since Λ is smooth, **Sard theorem of distance functions (Rifford (2004))** gives regular values $s_i \downarrow 0$. For these values, **Bangert (1982)** gives

$$\mathrm{reach}_\Omega(F_{s_i}) > 0.$$

The second distance level is $C^{1,1}$

Fix $F_i := F_{s_i}$ and set

$$r_i := \mathrm{reach}_\Omega(F_i) > 0.$$

At each $q \in \partial F_i$, the outward normal directions form a closed convex cone $\mathcal{N}_q F_i \subset T_q M$. Its unit vectors form the unit normal bundle $\mathcal{N}^1_\Omega F_i$. Federer's theorem (1959) parametrizes the tubular neighborhood of F_i by

$$\Phi : \mathcal{N}^1_\Omega F_i \times (0, r_i) \longrightarrow \{0 < d_{F_i} < r_i\}, \qquad \Phi(q, v, t) = \exp_q(tv).$$

Choose

$$0 < \tau < \min\{s_i, \tfrac{1}{2} r_i\}.$$

The slice

$$\Sigma_{i,\tau} := \{d_{F_i} = \tau\} = \left\{\exp_q(\tau v) : (q, v) \in \mathcal{N}^1_\Omega F_i\right\}$$

is a $C^{1,1}$ hypersurface. The tubular neighborhood also gives, almost everywhere,

$$-C(r_i) \le \kappa_j \le C(\tau).$$

The unique-normal case

Every smooth interior test hypersurface for Σ touches at a point of Λ. Thus $H_\Lambda \geq \delta$ means that Σ is a δ-barrier.
Fix $\delta' < \delta$ and choose i so that $Cs_i < \delta - \delta'$. Let D be a smooth region whose boundary touches $\Sigma_{i,\tau}$ from the inside at

$$p = \exp_q(\tau v), \qquad \mathcal{N}^1_q F_i = \{v\}, \qquad q_\Sigma = \exp_q(s_i v) \in \Sigma.$$

The unique normal shows that the same normal geodesic joins p to q_Σ. Set

$$r = s_i - \tau, \qquad D_r = \{x : \mathrm{dist}_g(x, D) < r\}.$$

Then ∂D_r touches Σ from the inside at q_Σ.
Along this parallel family, the Riccati equation gives

$$\frac{d}{dt} H_t = -|A_t|^2 - \mathrm{Ric}_g(\nu_t, \nu_t) \leq C, \qquad H_{\partial D_r}(q_\Sigma) \leq H_{\partial D}(p) + Cs_i.$$

If $H_{\partial D}(p) < \delta'$, then

$$H_{\partial D_r}(q_\Sigma) < \delta' + Cs_i < \delta,$$

contradicting the barrier property of Σ.

The multiple-normal case

Let ∂D touch $\Sigma_{i,\tau}$ from the inside at

$$p = \exp_q(\tau v).$$

The normal cone cannot contain both v and $-v$, since this would make q a critical point of d_Σ, contrary to the choice of s_i.
Choose another unit normal $v_* \not\parallel v$. Convexity of the normal cone gives

$$\zeta(\theta) = \exp_q\left(\tau \frac{v + \theta v_*}{|v + \theta v_*|_g}\right) \in \Sigma_{i,\tau}, \qquad \zeta(0) = p.$$

At the touching point, this curve and the two-sided ball condition give

$$\kappa_{\max}(\partial D, p) \geq \frac{1}{2\tau}, \qquad \kappa_{\min}(\partial D, p) \geq -C.$$

Hence, for large i,

$$H_{\partial D}(p) \geq \frac{1}{2\tau} - (n-2)C > \delta'.$$

Combining the two cases, $\Sigma_{i,\tau}$ is a $C^{1,1}$ $(\delta - o(1))$-barrier. It can then be smoothed (MCF, μ-bubble,...).

Lawson's problem for stationary boundaries

Let ∂E be stationary in a ball B. For every $B' \Subset B$ there are smooth $\Omega_j \Subset E \cap B$ such that

$$\partial\Omega_j \longrightarrow \partial E \quad \text{locally in } B', \qquad H_{\partial\Omega_j} > 0 \quad \text{on } \partial\Omega_j \cap B'.$$

Choose a concentric ball B_0 with $B' \Subset B_0 \Subset B$, transverse to $\mathcal{R}(\partial E)$, and set $\Omega = E \cap B_0$. The spherical part of $\partial\Omega$ has $H > 0$, and the transverse corner is not touched. Hence $\partial\Omega$ is quasi-regular with $H_\Lambda \geq 0$.
Set

$$u(x) = 1 + |x|^2, \qquad Ku := \inf_{|v|=1} \big(\Delta u - D^2 u(v,v)\big) = 2(n-1) > 0,$$

Following Ilmanen, a direct computation shows that, for small $r > 0$,

$$\Omega_r := \{x \in \Omega : d_{u^{-2} g_{\mathrm{Euc}}}(x, \partial\Omega) > r\}, \qquad H_{\Lambda(\Omega_r)} \geq \delta_r > 0.$$

Apply the mean convex smoothing theorem to Ω_r and then let $r \downarrow 0$.

Lawson (1986, Proc. Sympos. Pure Math.); Ilmanen (1996, Calc. Var.); Zhihan Wang (2024, GAFA).

Distance levels are already mean-convex

We now assume $\delta \geq 0$ and explain Gromov's argument. Set

$$K = \partial\Omega, \qquad \rho = \mathrm{dist}(\,\cdot\,, K).$$

The distance function ρ is locally semiconcave away from K. On $\{\rho > r\}$,

$$\nabla^2 \rho = A\,\mathcal{L}^n + S, \qquad A \leq r^{-1} I, \qquad S \leq 0.$$

At almost every x, quasi-regularity and Sard's theorem place x on a regular normal ray $x = p - t\nu(p)$ with $p \in \Lambda$. If $e = \nabla\rho(x)$ and κ_i are the principal curvatures at p, then

$$Ae = 0, \qquad \mu_i = -\frac{\kappa_i}{1 - t\kappa_i},$$

and therefore

$$\mathrm{tr}_{e^\perp} A = -H(p) - \sum_{i=1}^{n-1} \frac{t\kappa_i^2}{1 - t\kappa_i} \leq -\delta.$$

Bending

The estimate so far is only for the hyperplane $e^{\perp}$. Convolution averages points with different e, so this is not enough.

On the layer $U = \{|\rho - s| < h\}$, choose

$$L = \frac{n-2}{s-h} + \delta, \qquad \alpha'(s) = 1, \qquad \alpha'' = -L\alpha',$$

and set $w = \alpha \circ \rho$. Its Hessian is

$$\nabla^2 w = \alpha' \nabla^2 \rho + \alpha'' e \otimes e.$$

The first term retains the tangent-trace estimate. The second inserts one negative eigenvalue in the missing normal direction.

Indeed, for every rank-$(n-1)$ orthogonal projection P,

$$\mathrm{tr}(P\nabla^2 w) \leq \max\left\{-\delta\alpha', \left(\alpha'' + \frac{n-2}{s-h}\alpha'\right)\right\} \leq -\delta\alpha'$$

as a measure on U. This is the whole point of the bending.

Convolution

On a layer with $h = o(s)$, we have $\alpha' = 1 + o(1)$. Thus the preceding inequality becomes

$$\mathrm{tr}(P\nabla^2 w) \leq \big(-\delta + o(1)\big)\mathcal{L}^n \qquad \text{for every hyperplane projection } P.$$

It is linear and holds for every fixed P. Hence the mollification $w_\sigma = w * \varphi_\sigma$ satisfies

$$\mathrm{tr}(P\nabla^2 w_\sigma) \leq -\delta + o(1), \qquad |\nabla w_\sigma| \leq 1 + o(1).$$

Choose a regular value τ close to $\alpha(s)$ and set

$$E = \Omega \cap \{w_\sigma > \tau\}.$$

Then, after choosing σ sufficiently small,

$$\{\rho > s + h/2\} \subset E \subset \{\rho > s - h/2\}.$$

Taking P to be the tangent projection of ∂E gives

$$H_{\partial E} = -\frac{\mathrm{tr}(P\nabla^2 w_\sigma)}{|\nabla w_\sigma|} \geq \delta - o(1).$$

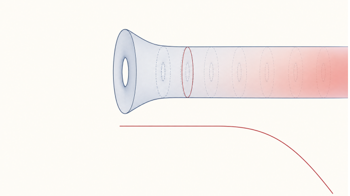
Finally let $s \downarrow 0$. The inclusions above give

$$d_H(\partial E, \partial\Omega) \to 0.$$

A soft barrier prevents escape

$$\liminf_{r\to\infty} \frac{\mathrm{vol}_g(B_r(p))}{r} = 0.$$

By coarea, choose $r_j \to \infty$ with $P(B_{r_j}(p)) \to 0$, where P denotes perimeter. For $K \Subset K_1$, direct perimeter minimization may escape to the outer cut $\partial B_{r_j}(p)$.
Choose $h = 0$ near K_1 and $h \ll 0$ near $\partial B_{r_j}(p)$, and minimize



$h = 0$ near K_1 and decreases near the outer cut.

$$\mathcal{A}^{h,\delta}(\Omega) = P(\Omega) - \int_\Omega (h - \delta)\, d\mu$$

Here $K_1 \subset \Omega \subset B_{r_j}(p)$.
The potential prevents escape, but the minimizer is no longer area-minimizing: its regular part satisfies $H = h - \delta \le -\delta$. Its boundary is quasi-regular, and the smoothing theorem gives $K \Subset U \Subset M$ with $H_{\partial U} < 0$.

Ilmanen (1996, Calc. Var.).

Toral ends have at least linear volume growth

For every complete metric g on $M^n = \mathbb{T}^{n-1} \times [0, \infty)$,

$$R_g \geq 0 \quad \Longrightarrow \quad \liminf_{r \to \infty} \frac{\mathrm{vol}_g(B_r(p))}{r} > 0.$$

There is no assumption on $H_{\mathbb{T}^{n-1} \times \{0\}}$.
If the lower limit vanished, the preceding result would give a smooth set U containing the original boundary. For $N = M \setminus U$, with the outward normal of N,

$$R_N \geq 0, \qquad H_{\partial N} > 0, \qquad \deg\big(\pi|_{\partial N} : \partial N \to \mathbb{T}^{n-1}\big) = 1.$$

A consequence of Lott's (spin) obstruction is

$$R_N \geq 0, \qquad H_{\partial N} > 0 \quad \Longrightarrow \quad \deg(\pi|_{\partial N}) = 0,$$

contradicting $\deg(\pi|_{\partial N}) = 1$.

Lott (2025, J. Reine Angew. Math., Section 2.2).

Penrose via Bray's conformal flow

Let (M^n, g) be asymptotically flat with $R_g \geq 0$, and let Σ_0 be its outermost minimal boundary. The target is

$$m_{\mathrm{ADM}}(M, g) \geq \frac{1}{2}\left(\frac{|\Sigma_0|_g}{n\omega_n}\right)^{\frac{n-2}{n-1}} .$$

Set $u_0 = 1$ and

$$g_t = u_t^{\frac{4}{n-2}} g, \qquad \Sigma_t = \text{the outermost } g_t\text{-minimizing enclosure of } \Sigma_0.$$

The velocity $v_t = \partial_t u_t$ is determined by

$$\begin{cases} \Delta_g v_t = 0 & \text{outside } \Sigma_t, \\ \quad v_t = 0 & \text{on and inside } \Sigma_t, \\ \quad v_t \longrightarrow -e^{-t} & \text{at infinity.} \end{cases}$$

Along the flow,

$$|\Sigma_t|_{g_t} = |\Sigma_0|_g, \qquad \text{the normalized limit is Schwarzschild.}$$

Hence the Penrose inequality follows once $m(t) := m_{\mathrm{ADM}}(M, g_t)$ is nonincreasing.

Bray (2001); Bray–Lee (2009).

Mass–capacity implies mass monotonicity

Let M_t be the unbounded exterior of Σ_t, and solve w_t:

$$\Delta_{g_t} w_t = 0, \qquad w_t|_{\Sigma_t} = 1, \qquad w_t \longrightarrow 0 \qquad \text{at infinity,}$$

$$c_t := c(\Sigma_t, g_t) = \frac{1}{n(n-2)\omega_n} \int_{M_t} |\nabla^{g_t} w_t|^2 \, d\mu_{g_t}.$$

For almost every t, Bray proved

$$m'(t) = -2\big(m(t) - c_t\big), \qquad \boxed{m(t) \geq c_t} \implies \boxed{m'(t) \leq 0}.$$

For $n \geq 8$, the flow boundary may be singular:

$$\Sigma_t = \mathcal{R}_t \cup \mathcal{S}_t, \qquad H_{\mathcal{R}_t} = 0, \qquad \dim_{\mathcal{H}} \mathcal{S}_t \leq n - 8.$$

The singular mass–capacity inequality is obtained by

$$(\Sigma_t, g_t) \xrightarrow{\text{perturbation}} (\widehat{\Sigma}, \widetilde{g}) \xrightarrow{\text{Gromov smoothing}} \Gamma_j \text{ smooth}, \qquad H_{\Gamma_j} > 0.$$

$$\Gamma_j \xrightarrow{\text{doubling + PMT with corners}} m_{\mathrm{ADM}}(\widetilde{g}) \geq c(\Gamma_j, \widetilde{g}) \xrightarrow{\text{limits}} m(t) \geq c_t.$$

Bray (2001); Bray–Lee (2009); Miao (2002).

Thank you for your attention